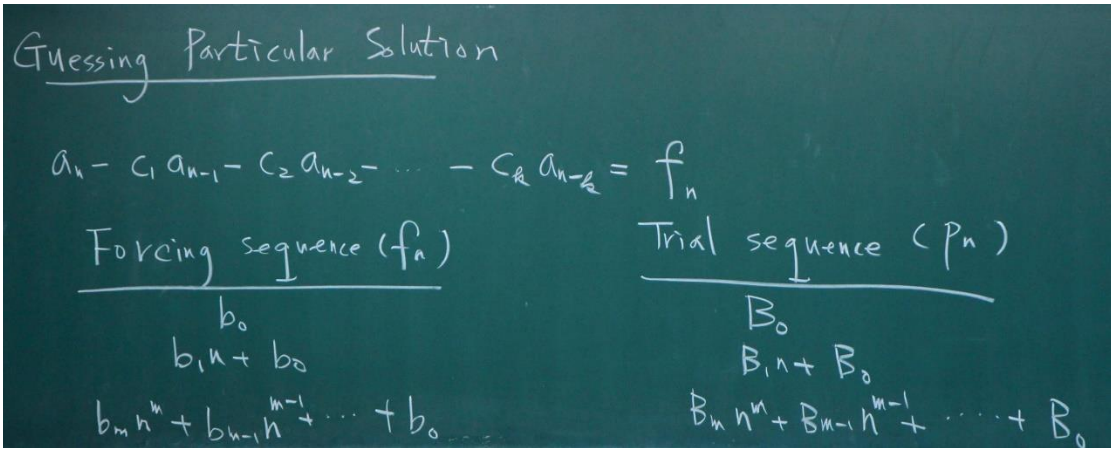
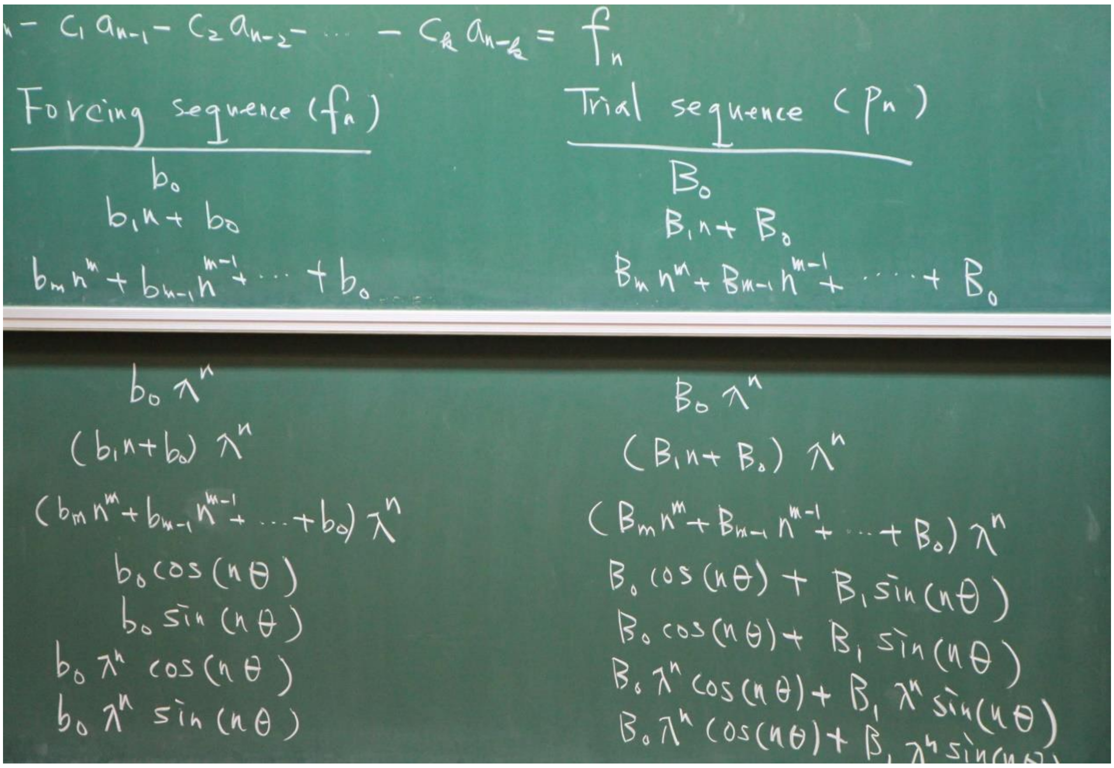
【10920 趙啟超教授離散數學 / 第 13 堂版書】





Guessing Particular Solution

$$a_n - c_1 a_{n-1} - c_2 a_{n-2} - \cdots - c_k a_{n-k} = f_n$$

Forcing sequence (f_n)	Trial sequence (p_n)
b_0	B_0
$b_1 n + b_0$	$B_1 n + B_0$
$b_m n^m + b_{m-1} n^{m-1} + \cdots + b_0$	$B_m n^m + B_{m-1} n^{m-1} + \cdots + B_0$
$b_0 \lambda^n$	$B_0 \lambda^n$
$(b_1 n + b_0) \lambda^n$	$(B_1 n + B_0) \lambda^n$
$(b_m n^m + b_{m-1} n^{m-1} + \cdots + b_0) \lambda^n$	$(B_m n^m + B_{m-1} n^{m-1} + \cdots + B_0) \lambda^n$
$b_0 \cos(n\theta)$	$B_0 \cos(n\theta) + B_1 \sin(n\theta)$
$b_0 \sin(n\theta)$	$B_0 \cos(n\theta) + B_1 \sin(n\theta)$
$b_0 \lambda^n \cos(n\theta)$	$B_0 \lambda^n \cos(n\theta) + B_1 \lambda^n \sin(n\theta)$
$b_0 \lambda^n \sin(n\theta)$	$B_0 \lambda^n \cos(n\theta) + B_1 \lambda^n \sin(n\theta)$

Example $a_n - 2a_{n-1} = n$, for $n \geq 1$
with $a_0 = 1$.

The general solution to the associated HRR is $\alpha_1 2^n$.

Let the trial sequence for a particular solution to the NRR be $p_n = B_1 n + B_0$.

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$$\Rightarrow \quad (B_1 n + B_0) - 2(B_1(n-1) + B_0) = n$$

$$\Rightarrow \quad -B_1 n + (2B_1 - B_0) = n$$

$$\Rightarrow \quad B_1 = -1, \quad B_0 = -2$$

$$\Rightarrow \quad p_n = -n - 2.$$

$\therefore$ The general solution to the NRR is

$$a_n = \alpha_1 2^n - n - 2.$$

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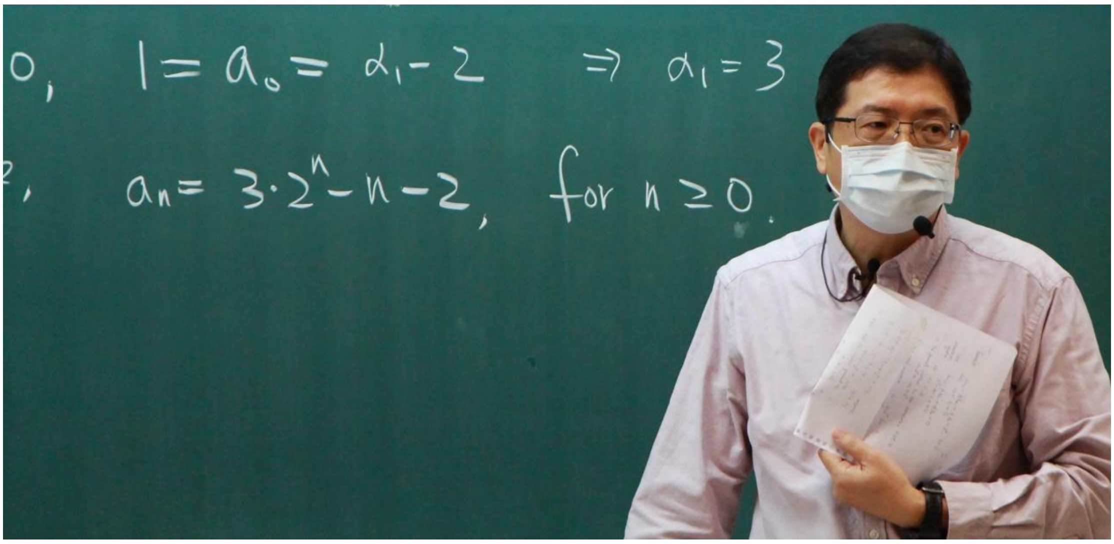
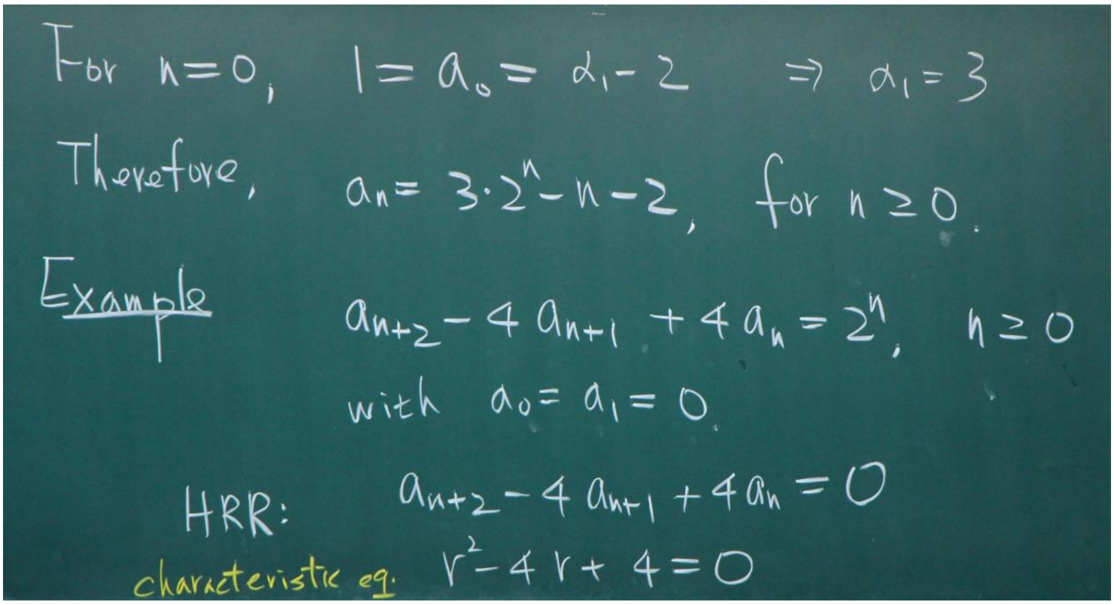
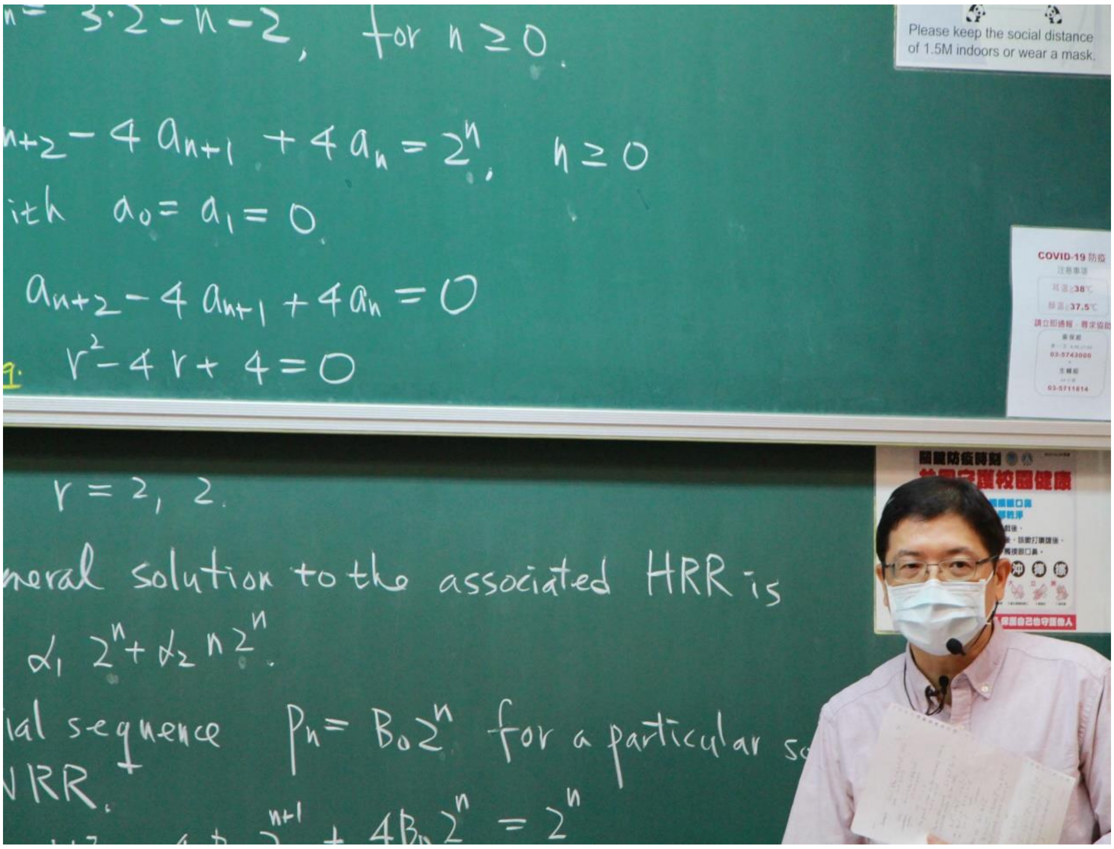

o, $1 = a_0 = \alpha_1 - 2 \Rightarrow \alpha_1 = 3$
$a_n = 3 \cdot 2^n - n - 2$, for $n \geq 0$.


For $n = 0$, $1 = a_0 = \alpha_1 - 2 \Rightarrow \alpha_1 = 3$
Therefore, $a_n = 3 \cdot 2^n - n - 2$, for $n \geq 0$.
Example
$a_{n+2} - 4a_{n+1} + 4a_n = 2^n$, $n \geq 0$
with $a_0 = a_1 = 0$.
HRR: $a_{n+2} - 4a_{n+1} + 4a_n = 0$
characteristic eq. $r^2 - 4r + 4 = 0$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r = 2, 2.
neral solution to the associated HRR is
α₁ 2ⁿ + α₂ n 2ⁿ.
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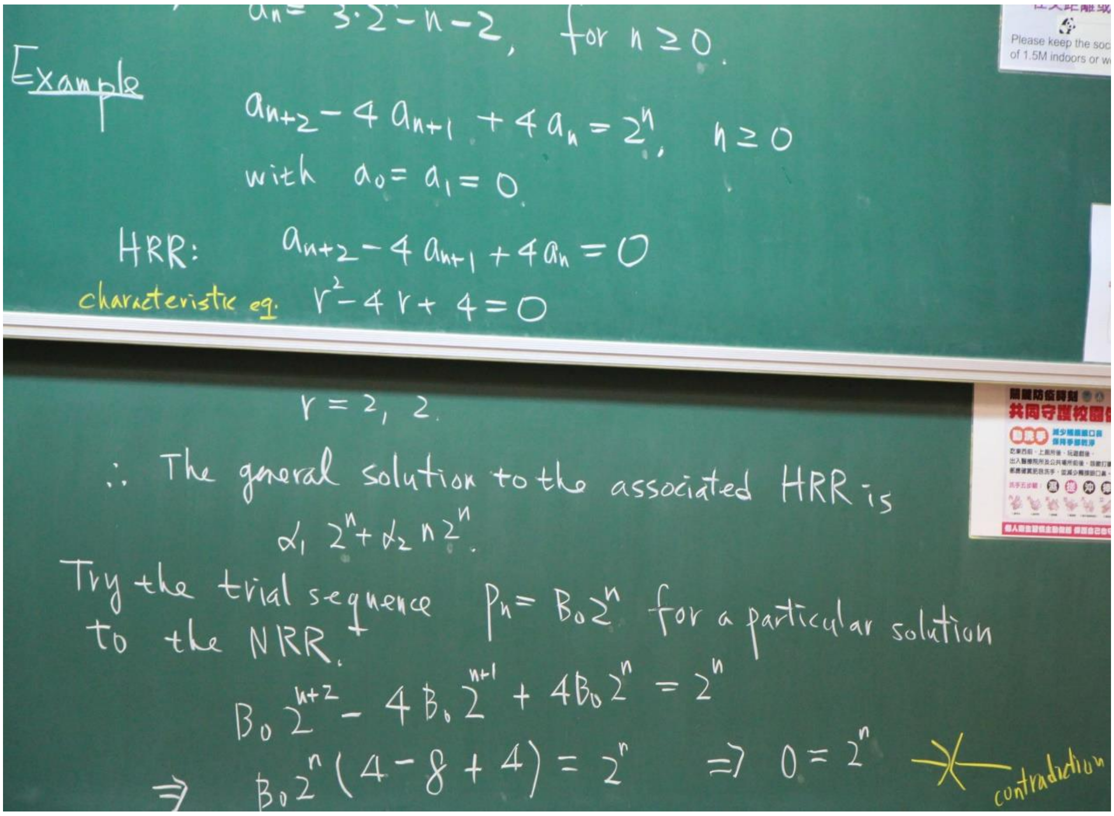
aₙ = 3·2ⁿ − n − 2, for n ≥ 0.
Example
aₙ₊₂ − 4 aₙ₊₁ + 4 aₙ = 2ⁿ, n ≥ 0
with a₀ = a₁ = 0.
HRR: aₙ₊₂ − 4 aₙ₊₁ + 4aₙ = 0
characteristic eq. r² − 4r + 4 = 0
r = 2, 2.
∴ The general solution to the associated HRR is
α₁ 2ⁿ + α₂ n 2ⁿ.
Try the trial sequence pₙ = B₀2ⁿ for a particular solution
to the NRR.
B₀ 2ⁿ⁺² − 4 B₀ 2ⁿ⁺¹ + 4B₀ 2ⁿ = 2ⁿ
⇒ B₀ 2ⁿ (4 − 8 + 4) = 2ⁿ ⇒ 0 = 2ⁿ
contradiction
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For $n=0$, $\quad 1 = a_0 = \alpha_1 - 2 \quad \Rightarrow \alpha_1 = 3$

Therefore, $\quad a_n = 3\cdot 2^n - n - 2$, for $n \ge 0$.

Example

$$a_{n+2} - 4a_{n+1} + 4a_n = 2^n, \quad n \ge 0$$

with $a_0 = a_1 = 0$.

HRR: $\quad a_{n+2} - 4a_{n+1} + 4a_n = 0$

characteristic eq. $\quad r^2 - 4r + 4 = 0$

$$r = 2, 2$$

$\therefore$ The general solution to the associated HRR is

$$\alpha_1 2^n + \alpha_2 n 2^n.$$

Try the trial sequence $p_n = B_0 2^n$ for a particular solution to the NRR.

$$B_0 2^{n+2} - 4B_0 2^{n+1} + 4B_0 2^n = 2^n$$

$$\Rightarrow B_0 2^n (4 - 8 + 4) = 2^n \quad \Rightarrow 0 = 2^n \quad \rightarrow\leftarrow \text{ contradiction}$$

Then try $p_n = B_0 n 2^n$.

$$B_0 (n+2) 2^{n+2} - 4B_0 (n+1) 2^{n+1} + 4B_0 n 2^n = 2^n$$

$$\Rightarrow B_0 2^n (4n + 8 - 8n - 8 + 4n) = 2^n$$

$$\Rightarrow 0 = 2^n \quad \rightarrow\leftarrow$$

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Now try $p_n = B_0 n^2 2^n$

$$B_0 (n+2)^2 2^{n+2} - 4 B_0 (n+1)^2 2^{n+1} + 4 B_0 n^2 2^n = 2^n$$

$$\Rightarrow \quad B_0 2^n \left[4(n+2)^2 - 8(n+1)^2 + 4n^2 \right] = 2^n$$

$$\Rightarrow \quad B_0 2^n (4n^2 + 16n + 16 - 8n^2 - 16n - 8 + 4n^2) = 2^n$$

$$\Rightarrow \quad 8 B_0 2^n = 2^n$$

$$\Rightarrow \quad B_0 = \frac{1}{8}.$$

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$\therefore$ $p_n = \frac{1}{8} n^2 2^n$ is a particular solution to the NRR.

Hence the general solution to the NRR is

$$a_n = \alpha_1 2^n + \alpha_2 n 2^n + \frac{1}{8} n^2 2^n.$$

For initial conditions,

$$0 = a_0 = \alpha_1$$

$$0 = a_1 = 2\alpha_1 + 2\alpha_2 + \frac{1}{4}$$

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$$\Rightarrow \quad \alpha_1 = 0, \quad \alpha_2 = -\frac{1}{8}.$$

$$\therefore \quad a_n = -\frac{1}{8} n 2^n + \frac{1}{8} n^2 2^n$$

$$= n(n-1)\, 2^{n-3} \quad \text{for } n \geq 0.$$

In general, if the trial sequence mentioned above fails, multiply it by n. Try again. Repeat this procedure as often as necessary to find a particular solution.

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